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Abstract

Numerical models that simulate the dynamics of carbon in soil are increasingly used to improve our knowledge and help our management of the carbon cycle. Calculation of the long term behavior of these models is necessary in many applications but encounters the difficulty of managing the periodic forcing variables, e.g., seasonal variations, such as carbon inputs and decomposition rates. This calculation is conventionally done by running the model over large time durations or by assuming constant forcing variables. Two methods, which make it possible to rapidly compute periodic solutions taking into account the time variations of these variables, are proposed. The first one works on discrete-time models and the second one on continuous-time models involving Fourier transforms. Both methods were tested on the Rothamsted carbon model (RothC), a discrete-time model which has also been given a continuous approximation, using realistic an unrealistic sets of time-varying forcing functions. Both methods provided an efficient way to compute the periodic solutions of the RothC model within the application domain of the model. Compared to running the discrete model to the equilibrium, reduction in the computational cost was of up to 95% at the expense of a maximum absolute error of 1% for the estimation of carbon stocks. For specific distributions of the forcing variables the use of Fourier transform of zero order, which was equivalent to assume constant forcing variables, led to a maximum absolute error of 55% in the estimation of the long term behavior of the model. There, a Fourier transform of order higher than zero is required.

Keywords: soil organic carbon dynamics, discrete formulation, continuous formulation, steady state, periodic solutions, linear model, Fourier series.

Introduction

1.1. carbon dynamics

The soil organic carbon (SOC) plays an important role in several environmental and land management issues. One of the most important issues is the role that SOC plays as part of the terrestrial carbon and might play as a regulator of the atmospheric CO2. Many factors are likely, in a near future, to modify the SOC content, including changes in agricultural practices (Betts, 2000; Vleeshouwers and Verhagen, 2002; Bellamy et al., 2005) and global climate changes (Jenkinson et al., 1991; Cao and Woodward, 1998; Cox et al., 2000; Jones et al., 2005; Knorr et al., 2005). Understanding SOC and soil organic matter (SOM) dynamics as a function of soil characteristics, agricultural management and climatic conditions is therefore crucial, and many models have been developed in this perspective. Most models of SOM turnover, excepting a few (Bosatta and Agren, 2003), are compartmental models, exhibiting various degree of complexity. The compartments represent carbon originating from plants or contained in soil and transformed by microorganisms and each one is characterized by a particular decomposition rate representing more labile or more stable forms of soil organic matter. Some models include N turnover and/or plant growth modules (CENTURY) when others only focus on SOC (RothC). Also, most use a linear method of transferring quantities between the different compartments (Baisden and Amundson, 2003) but some models including non-linear dynamics have also been developed more recently (Manzoni et al., 2004).

These models are used in a variety of ways and often for long term studies (Coleman et al., 1997; Falloon and Smith, 2002; Franko et al., 2002; Shevtsova et al., 2003; Shirato, 2005; Shirato et al., 2005b; Shirato and Yokozawa, 2005a). The behavior of the SOC system, over a long term and assuming that the environment of the system (inputs of organic carbon, climatic conditions) is stable, is reported to tend toward a steady state. Although many soils under study might not have reached equilibrium, being able to compute and predict the long-term solution is extremely valuable. It gives a synthetic view of the system in given agro-climatic conditions, makes it possible to test if a studied soil has reached an equilibrium or not, to envision what would be the consequences of specific events onto a given soil assuming that a new stable state is reached and to serve as a control case or initial conditions (Thornton and Rosenbloom, 2005). Technically, the equilibrium assumption is also commonly used to solve analytically mathematical systems. Such an analytical solution gives an explicit relationship between model inputs and outputs and may, in turn, be computed without simulating or integrating numerically the system until it reaches a stable state, thus saving computation time. When models cannot be formulated analytically, estimating the steady state solutions still can be useful and generic or model specific efficient numerical methods are available (Thornton et al., 2005). More generally, being able to use analytical forms of the long-term solutions is particularly useful in understanding models behavior and relationships between input and output variables of the model.

Some of the SOC models have been formulated mathematically (Parshotam, 1996; Bolker et al., 1998; Yang et al., 2002; Baisden et al., 2003; Manzoni et al., 2004) and approaches, as the development of the ICBM family models (Katterer and Andren, 2001), specifically aim at proposing analytically solved models representing the conventional wisdom of soil C and N modelling. For these models, when studying N and C soil content at steady state, it is usually assumed that forcing variables (typically climatic variables and variables representing inputs) can be set to their average value, calculated over a representative year for instance, which considerably eases the mathematical treatment. There, the long-term behavior of the model truly is a steady state. Consequences of such an assumption have for now been tested only empirically for some models and specific conditions. In some cases, environmental shifts from one stable state to another or brief events are considered and mathematical treatment used to estimate the new stable state after perturbation or the system resilience. We propose here two methods which make it possible to deal with continuously time varying agro-climatic conditions (e.g. forcing variables), when they can be specified as periodic functions. The first one works on discrete-time models and the second one on continuous-time models involving Fourier transforms. We considered the Rothamsted model with crop cultivations, as representative of many models of soil organic matter dynamics, anticipating that the second method could also be applied to models involving non-linear dynamics. We used these methods to test the consequences of assuming yearly constant agro climatic condition instead of considering their intra-annual variability.

Methods

1.2. Discrete formulation

The RothC model (Coleman et al., 1997) splits the soil carbon into four active compartments and one inactive. At each time step, the four active compartments, decomposable plant material, DPM, resistant plant material, RPM, the microbial community BIO and the humus, HUM, undergo decomposition as a function of a rate constant, depending on the compartment and on a rate modifier. The rate modifier depends on the clay content of the soil, climatic variables and land cover. Products of the decomposition are CO_2 and carbon feeding the BIO and HUM compartments. The fraction of the decomposed carbon incorporated into BIO and HUM increases as a function of the clay content of the soil. Carbon enters the soil through the DPM and RPM compartments. The fraction input in DPM and RPM respectively is chosen as a constant which is an estimate of the decomposability of the plant material. It depends on the cultivation being considered. The model can be formulated as

$$C_{t+1} = F_t C_t + B_t$$

Where

$$C_t = \begin{pmatrix} dpm \\ rpm \\ bio \\ hum \end{pmatrix}_t$$

$$F_t = \begin{pmatrix} e^{-\rho_t k_{dpm}} & 0 & 0 & 0 \\ 0 & e^{-\rho_t k_{rpm}} & 0 & 0 \\ \alpha(1-e^{-\rho_t k_{dpm}}) & \alpha(1-e^{-\rho_t k_{rpm}}) & \alpha(1-e^{-\rho_t k_{bio}}) + e^{-\rho_t k_{bio}} & \alpha(1-e^{-\rho_t k_{hum}}) \\ \beta(1-e^{-\rho_t k_{dpm}}) & \beta(1-e^{-\rho_t k_{rpm}}) & \beta(1-e^{-\rho_t k_{bio}}) & \beta(1-e^{-\rho_t k_{hum}}) + e^{-\rho_t k_{hum}} \end{pmatrix}$$

and

$$B_t = \begin{pmatrix} a_{dpm} \\ a_{rpm} \\ a_{bio} \\ a_{hum} \end{pmatrix} b_t$$

The four input coefficients (a_{dpm}, a_{rpm}, a_{bio} and a_{hum}) sum up to 1 and in the most common case one uses $a_{dpm}=\gamma$, $a_{rpm}=1-\gamma$, $a_{bio}=0$ and $a_{hum}=0$, γ depending on the quality of the plant material.

Here, α and β are fractions of metabolized C incorporated respectively into BIO and HUM. b_t is the carbon amount ($t.ha^{-1}$) entering the system at month t, k_i the decomposition rate for compartment i and ρ_t the rate modifier.

Characterization of the long-term behavior

In case where b_t and ρ_t are constant, one can demonstrate that the (I_4-F) matrix, where the matrix I_4 is the 4-by-4 identity matrix, has an inverse (see below) and that the system yields a steady state solution. Assuming that F and B are respectively the time constant carbon flows and carbon inputs, one can write

$$C^* = (I_4 - F)^{-1} B$$

However, usually F_t and B_t vary through time but it can be assumed that they have a periodic behavior. Typically, if the agronomical practices are cyclic and if the weather conditions can be considered as periodic, ρ_t, b_t, and consequently F_t, B_t will also behave periodically. Assuming that the periodicity of these variables is P, one looks for a solution of C such that

$$C_{t+P} = C_t \tag{1}$$

For example, considering the common case the case where P is 12 months, we can write down

$$\begin{aligned} C_{t+1} &= F_1 C_t + B_1 \\ &.. \\ C_{t+11} &= F_{11} C_{t+10} + B_{11} \\ C_t &= F_{12} C_{t+11} + B_{12} \end{aligned}$$

which can be reformulated as:

$$\begin{pmatrix} 0 & I_4 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I_4 \\ I_4 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} C_t \\ \vdots \\ \vdots \\ C_{t+10} \\ C_{t+11} \end{pmatrix} = \begin{pmatrix} F_1 & 0 & \cdots & \cdots & 0 \\ 0 & F_2 & \ddots & & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & F_{12} \end{pmatrix} \begin{pmatrix} C_t \\ \vdots \\ \vdots \\ C_{t+10} \\ C_{t+11} \end{pmatrix} + \begin{pmatrix} B_1 \\ \vdots \\ \vdots \\ B_{11} \\ B_{12} \end{pmatrix} \tag{2}$$

and finally yields:

$$\begin{pmatrix} C_t \\ \vdots \\ \vdots \\ \vdots \\ C_{t+11} \end{pmatrix} = -\begin{pmatrix} F_1 & -I_4 & 0 & \cdots & 0 \\ 0 & F_2 & -I_4 & & \cdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -I_4 \\ -I_4 & 0 & \cdots & 0 & F_{12} \end{pmatrix}^{-1} \begin{pmatrix} B_1 \\ \vdots \\ \vdots \\ \vdots \\ B_{12} \end{pmatrix} \tag{3}$$

Solving this system (which can be performed *via* matrix inversion by common statistical packages or spreadsheet programs) yields a vector of dimension 4*12, which is the sequence of states $C_i=(dpm_i, rpm_i, bio_i, hum_i)^T$, i in $|[1,12]|$ and which characterizes the oscillatory state of carbon stock in each compartment, keeping track of the temporal variability of the forcing variables over the period.

1.3. Continuous formulation

In real soil systems, processes involved in the RothC model are continuous in time and thus one can propose the following continuous formulation, where C' denotes the derivative of C with respect of time:

$$C'(t) = \rho(t)AC(t) + B(t) \tag{4}$$

with

$$A = \begin{pmatrix} -k_{dpm} & 0 & 0 & 0 \\ 0 & -k_{rpm} & 0 & 0 \\ \alpha k_{dpm} & \alpha k_{rpm} & (\alpha-1)k_{bio} & \alpha k_{hum} \\ \beta k_{dpm} & \beta k_{rpm} & \beta k_{bio} & (\beta-1)k_{hum} \end{pmatrix} \tag{5}$$

$$B(t) = \begin{pmatrix} a_{dpm} & a_{rpm} & a_{bio} & a_{hum} \end{pmatrix}^T b(t)$$

$\rho(t)$ is the decomposition rate modifier. In the current RothC formulation, $\rho(t)$ is a function of monthly rainfall, temperature, pan open evaporation and land cover, as well as percentage of clay in the considered soil. When the time varying input variables (climatic and agricultural variables) are considered as periodic functions of period P, $\rho(t)$ itself is a periodic function with the same period. $b(t)$ is considered on a periodical basis too, as for the discrete formulation of the model. Again, the system defined by $C(t)$ is expected to tend toward a oscillatory state as $t \to +\infty$. Study of the eigenvalues of $\rho(t)A$ enables to characterize such a behavior. Let us define $y(t)$ and $\mathcal{A}(t)$.

$$y(t) = e^{-\mathcal{A}(t)}C(t) \tag{6}$$

where $\mathcal{A}(t)$ is the primitive of $\rho(t)A$.
From Eq.(4) and (6) we obtain

$$y'(t) = e^{-\mathcal{A}(t)}B(t)$$

and

$$y(t) = y_0 + \int_0^t e^{-\mathcal{A}(s)}B(s)ds \tag{7}$$

Setting $\mathcal{A}(0)$ to the four by four zero matrix, from Eq.(6) and (7) it can be written

$$C(t) = e^{A(t)}C_0 + e^{A(t)}\int_0^t e^{-A(s)}B(s)ds$$

We can first observe that if $A(t)$ eigenvalues are negative, $C(t)$ as $t \rightarrow +\infty$ do not depend on initial conditions C_0. Secondly, if $C(t)$ has a periodic solution, say C_0, with period P, it can be shown using semigroup representations that it satisfies the following Equation,

$$C_0 = e^{A(T)}C_0 + e^{A(T)}\int_0^T e^{-A(s)}B(s)ds$$

A solution to this equation exists and is unique if (Id-$e^{-A(t)}$) has an inverse, which can be demonstrated to be true if A is invertible. A sufficient condition for A to be invertible is that all its eigenvalues are nonpositive, i.e. using the definition of A (Eq.(5)) when $\alpha+\beta<1$. This condition is always true for the RothC model due to the definition of α and β (Coleman and Jenkinson, 1995) and thus the stock of carbon in each compartment tends towards a periodic solution in large times whatever the input values are.

Characterization of the long-term behavior

Approximations of this behavior can be made using Fourier series. Setting

$$C_N(t) = \sum_{k=-N}^{N} C_k e^{ikt}\ ,\ \rho_N(t) = \sum_{k=-N}^{N} \rho_k e^{ikt} \text{ and } b_N(t) = \sum_{k=-N}^{N} b_k e^{ikt} \tag{8}$$

with

$$\rho_k = \int_0^{2\pi} \rho(s)e^{-iks}ds \text{ and } b_k = \int_0^{2\pi} b(s)e^{-iks}ds$$

ρ_k and b_k coefficients can be obtained from the monthly input values used in the RothC model. $C(t)$, $A(t)$ and $B(t)$ can be replaced by their respective Fourier transform in Eq.(4) giving the following approximation.

$$\sum_{k=-N}^{N} ikC_k e^{ikt} = A \sum_{j,k/j+k=-N}^{N} C_j \rho_k e^{i(j+k)t} + \sum_{k=-N}^{N} b_k e^{ikt}$$

Setting N, the order of the Fourier series, to zero, we calculate the C_0 term.

$$C_0 \quad = \quad -A^{-1}B_0 / \rho_0 \tag{9}$$

Assuming that $B_0 = (\gamma,\ 1\text{-}\ \gamma,\ 0,\ 0)^T.b_0$, which means that carbon inputs only to the DPM and RPM compartments, leads to the following solution:

$$C_0 = \begin{pmatrix} \gamma b_0 / \rho_0 k_{dpm} \\ (1-\gamma) b_0 / \rho_0 k_{rpm} \\ b_0 \alpha / [(1-\alpha-\beta)\rho_0 k_{bio}] \\ b_0 \beta / [(1-\alpha-\beta)\rho_0 k_{hum}] \end{pmatrix} \qquad (10)$$

b_0 and ρ_0 terms represent averages of $b(t)$ and $\rho(t)$ over the considered period, which is one year. This solution gives an explicit formulation of the long-term behavior of the system, which obviously equals what would have been found using the assumption of constant forcing variables set to the average values. This solution does not enable to takes into account the temporal variability of the carbon stocks throughout the year. This could have been achieved by computing the C_1 term which itself is an approximation of the primary oscillations of the system's long-term behavior. Such a calculation lies beyond the scope of this paper. These oscillations directly depend on the temporal variability of the forcing variables but their size is usually small compared to the total SOC. In the following, they will be handled only with the discrete formulation of the RothC model (Eq.(3)).

1.4. Comparison of the approaches

Parshotam (1996) showed that given some restrictions the continuous formulation (Eq.(4)) is a good approximation of what would be the continuous formulation of the original discrete time RothC model. It is possible to turn this the other way round and say that the RothC model is an approximation of the discretization of the continuous model given in Eq.(4). Discretization of Eq.(4) leads to, in case of constant inputs during the sampling intervals (Parshotam, 1996):

$$C_{(k+1)\Delta t} = e^{\rho(k\Delta t)A\Delta t} C_{k\Delta t} + (\rho(k\Delta t)A)^{-1}(e^{\rho(k\Delta t)A\Delta t} - I_4)B(k\Delta t)$$

RothC is an approximation of the above equation because

$$F_t \approx e^{\rho(k\Delta t)A\Delta t} \quad \text{and} \quad B_t \approx (\rho(k\Delta t)A)^{-1}(e^{\rho(k\Delta t)A\Delta t} - I_4)B(k\Delta t)$$

Thus, the parameters used in the continuous model (e.g. α, β, k_{dpm}, k_{rpm}, k_{bio} and k_{hum}) should not be equated with those of the RothC model. However, numerically it makes little difference, and in the following developments, we shall do it.

Both approaches (using the discrete or the continuous formulation) can be used to characterize the periodic long-term behavior of the system. The first approach (Eq.(3)) uses the discrete formulation of the model which formally reproduces the specification of the RothC model contrary to the second one (Eq.(9)) which uses a continuous formulation and a zero order Fourier transform. This latest approach gives a simpler explicit solution, function of the input variables and parameters of the model (α, β, k_{dpm}, k_{rpm}, k_{bio} and k_{hum}). It might also be more interesting to work with the continuous form of the model because of its greater generality and since it is usually easier to discretize a continuous model rather than doing the opposite.

To assess the validity of both approaches in characterizing the long-term behavior with time-varying forcing variables, we first compared both approaches between themselves and with the discrete model where variability of forcing is leveraged over the year. This was performed using a weather dataset composed by monthly averages calculated over 12 years (1992-2004) on a 0.125° grid (4144 cells) covering the French country. For each point of the grid, we considered a unique crop system, with inputs being 0.50, 0.20, 0.10, 0.10, 0.10, 1.44 $tC.ha^{-1}$ respectively for each month from March to August, otherwise null with a bare soil (adapted from Swinnen et al., 1995 and Bolinder et al., 1997 for winter wheat). %Clay was set to 10%. The computation of the long-term behavior using the discrete formulation resulted in a periodic solution, i.e. a sequence of twelve C_i states, i in [1..12] (Eq.(3)). The solution given by a discrete formulation with averaged forcing variables was a single state noted C_{avg} , the calculation obtained using the continuous formulation was also a single state, noted C_0 (Eq.(9)). We compared C_{avg}, with C_0 and with the average state of the C_i states, noted $\langle C_i \rangle$.

To test more systematically the validity of the estimator based on Fourier series introduced in (Eq.(8)), e.g. C_0, the periodic solutions were also computed when varying the forcing variables independently, using in some case extreme and unrealistic values. The precision of C_0 was assessed using the σ/C_0 ratio, where σ is the standard deviation of the C_i states and represents the size of the oscillations characterizing the periodic solution. The bias of C_0 was assessed using the abs$(\langle C_i \rangle-C_0)/C_0$ ratio. We modeled the distributions of the forcing variables using Gaussian functions as

$$y_i = y_{\min} + \frac{N_i(d)}{\sum_i N_i(d)} a \quad \text{with} \quad N_i(d) = \frac{1}{d\sqrt{2\pi}} e^{-\frac{(i-6)^2}{2d^2}} \tag{11}$$

Where y_i, i in [1..12] is the value of the forcing variable (either $b(t)$ or $\rho(t)$) at month i, a and d respectively the amplitude and dispersion characterizing the distributions, and y_{min} a minimum value for the considered variable. Low values of the d parameter yielded distributions having a spike around the sixth month, high values uniform distributions. The a parameter represented a scaling parameter. a and d were varied at once and sampled linearly within their range of variation. y_{min} and the range of variation of d and a were, respectively for $b(t)$ and

$\rho(t)$, (0.1, [0.1,5] and [0.1,10]) and (0.83, [0.1,5] and [0.1,100]). One forcing variable remained constant whilst the amplitude and dispersion of the other was varied and set to *0.2* for b_i, i in [1..12] and set to 0.8, 0.8, 0.8, 0.82, 1.06, 1.10, 1.06, 0.82, 0.8, 0.8, 0.8, 0.8 for ρ_i, i in [1..12].

To compute the solutions with both approaches, the calculation of modifiers of the decomposition rates was done using SQL requests under the PostgreSQL DBMS and further calculations using the R Software. All the computations were done on a bi-xeon, 2Go RAM.

Results

Computation times on the standard climatic dataset were 57.9", and 46.9" for respectively the *Ci* (disctrete formulation) and C_0 (continuous formulation) calculations (to be compared to 15'10" needed when running the Fortran implementation of RothC available online at the Rothamsted Research website; this time includes for each point of the climatic dataset reading the data files, running the model using the equilibrium mode and writing the results).

On the long term, temporal variability of the forcing variables resulted in oscillations of the total SOC which size reached for some points of the standard climatic dataset 11% of the total SOC, as estimated using the discrete formulation. However, over this standard climatic spectrum, all methods gave similar results regarding the average SOC value at equilibrium (maximum absolute error of 1%). The long term values for the different carbon compartments at equilibrium were all similar but the value for the DPM compartment (Figure 1). For this compartment, the effect of the temporal variability of the forcing variable (obtained by comparing $<C_i>$ to the other solutions) was important. The error caused by using RothC as the discretization of the continuous formulation (see § 1.4) can be seen when comparing C_{avg} with C_0. It appears that has C_{avg} slightly overestimates the DPM pool compared to C_0.

C_0 became strongly biased and imprecise for extreme distributions of $\rho(t)$ where amplitude of this forcing variable was high and variability over the period was large (for a=100 and c=0.1 yielding maximum ρ values of 100.1, bias reached 0.5 and imprecision 0.27, Figure 2, top diagrams). The imprecision and bias of C_0 also depended on the amplitude and dispersion of $b(t)$ but always remained small (Figure 2, bottom diagrams). We checked (not displayed here) that the bias and imprecision of C_0 compared to C_i was not caused by using RothC as a discretization of the continuous model but by the fact that C_0 leverages the time-variability of the forcing variables when C_i does not. Thus, the domains where C_0 is imprecise and more importantly biased are the domains where the time-variability of the forcing variables greatly determines the behavior of the system.

Conclusion

The continuous formulation using Fourier transforms makes it possible to specify analytically the forcing variables as functions of time, and then to obtain analytical solutions for the mathematical formulation of the model of carbon dynamics under study. Here, we used a zero order transform, which makes the forcing variables constant through time, and studied the validity of such an assumption.

We showed in turn that such an approximation resulted in short computation times and is reasonably precise (i) for the common application domain of the RothC model and (ii) in case there is no concern about intra-annual variations of decomposable plant material and to a smaller extent of the microbial community. If these conditions are not met, one may want to use the discrete formulation or higher order Fourier transforms in order to grasp more of the temporal variability of the variables. It is likely that in standard conditions, the use of average agro-climatic conditions for computing steady state solutions of linear models of organic matter dynamics, which is commonly found in literature on the subject (Bolker et al., 1998; Baisden et al., 2003; Manzoni et al., 2004), can be legitimated. The Fourier series approach is not restricted to linear models or to models taking only C dynamics into account. It could be particularly relevant for non-linear systems, where assuming that forcing variable can be averaged could become even more tedious than for linear systems. We also emphasize here the fact that the method proposed to deal with the continuous formulation, since it essentially relies on the use of Fourier series, is most suited to the modeling of periodic functions and to the case where decomposition and input functions can themselves be considered as periodical.

The approach concerning discrete-time models was used here as a way to quickly compute the long-term behavior of the discrete-time model RothC, without making any approximations and thus having a larger application domain than the approach using the continuous formulation. It is not likely to result in a simple analytical formulation of the equilibrium of the system, because of the relative complexity of the matrix to be inversed in order to compute the solution. Nevertheless, it speeds up considerably the computation compared to the use of the RothC's implementation. It can be applied to more complex systems and makes it possible to take into account the full time-variability of the discrete forcing variables. There might be constraints on the applicability of the discrete method, aimed at ensuring the solvability of the system described in Eq.(3). Determining these constraints is out of the scope of this paper but results about the Toeplitz matrices might help in this perspective. Indeed, the algebraic structure of system (2) is of circulant form (or Toeplitz) and this allows the use of very efficient methods for solving the eigenvalues problem we are interested in (Gray, 2006).

The methods proposed here to compute equilibrium solutions gave, within the RothC application domain, similar results for long term SOC dynamics compared with the Fortran implementation of the RothC model, while being up to 19.4 times faster. This might be critical when applying the model on very large data sets, for instance those produced by combining climatic, landuse and soil characteristics layers within a GIS, in order to spatially compute long term SOC stocks. In addition, working at equilibrium simplifies the analysis of the results as only the long-term solution is considered. Both methods are of course not restricted to one-year periods and could be applied to cycles with much longer periods, for instance, to crop rotations or to large climatic oscillations (with periods). It would be also interesting to consider the case where, while remaining oscillatory, the forcing variables exhibit a drift. This could be applied, for instance, to study the effect of climate change on SOC stocks.

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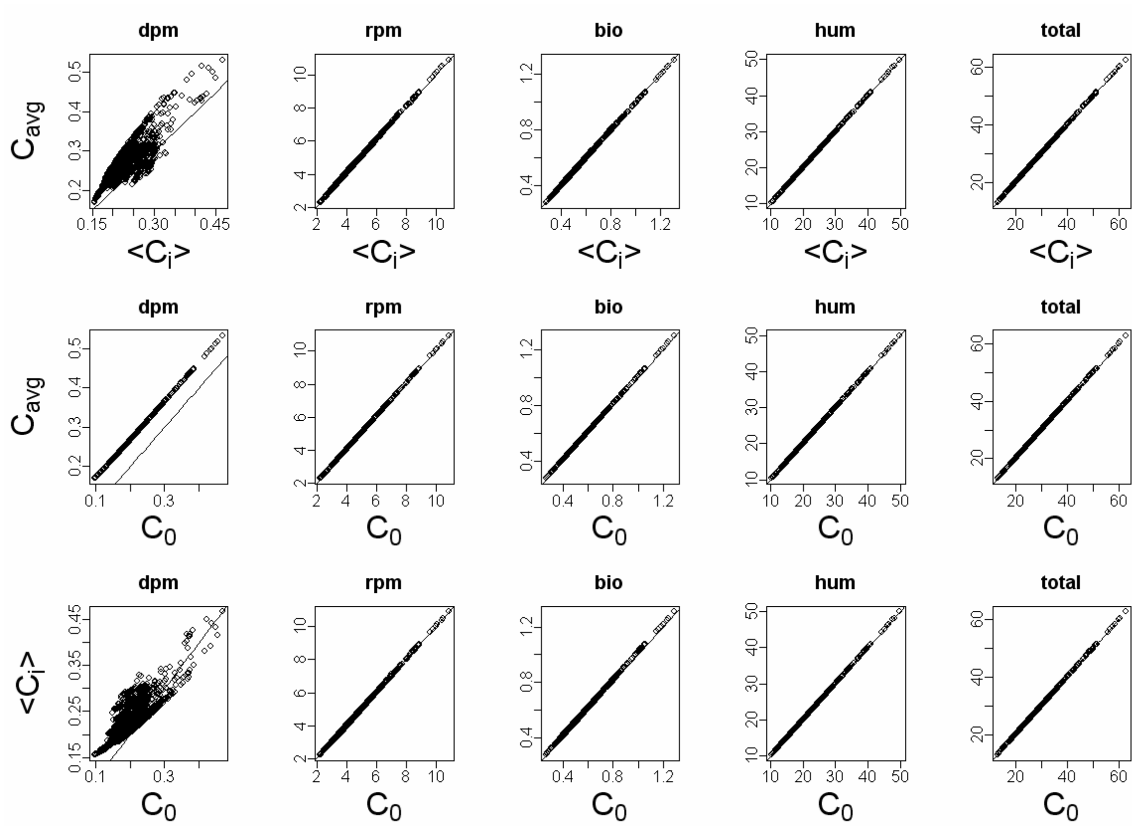

Figure 1 : long-term solutions for each compartment of the model and for the total carbon content, over the whole set of climatic conditions and with %clay=10%. All values are given in t.ha^{-1}. First line of diagrams plots the results of the discrete formulation of RothC ($<C_i>$) against the results obtained with the discrete formulation with constant forcing variables (C_{avg}). Line Two gives the results of the continuous form (C_0) against C_{avg} and line three $<C_i>$ against C_0. The plain lines drawn on the charts represent the y=x function.

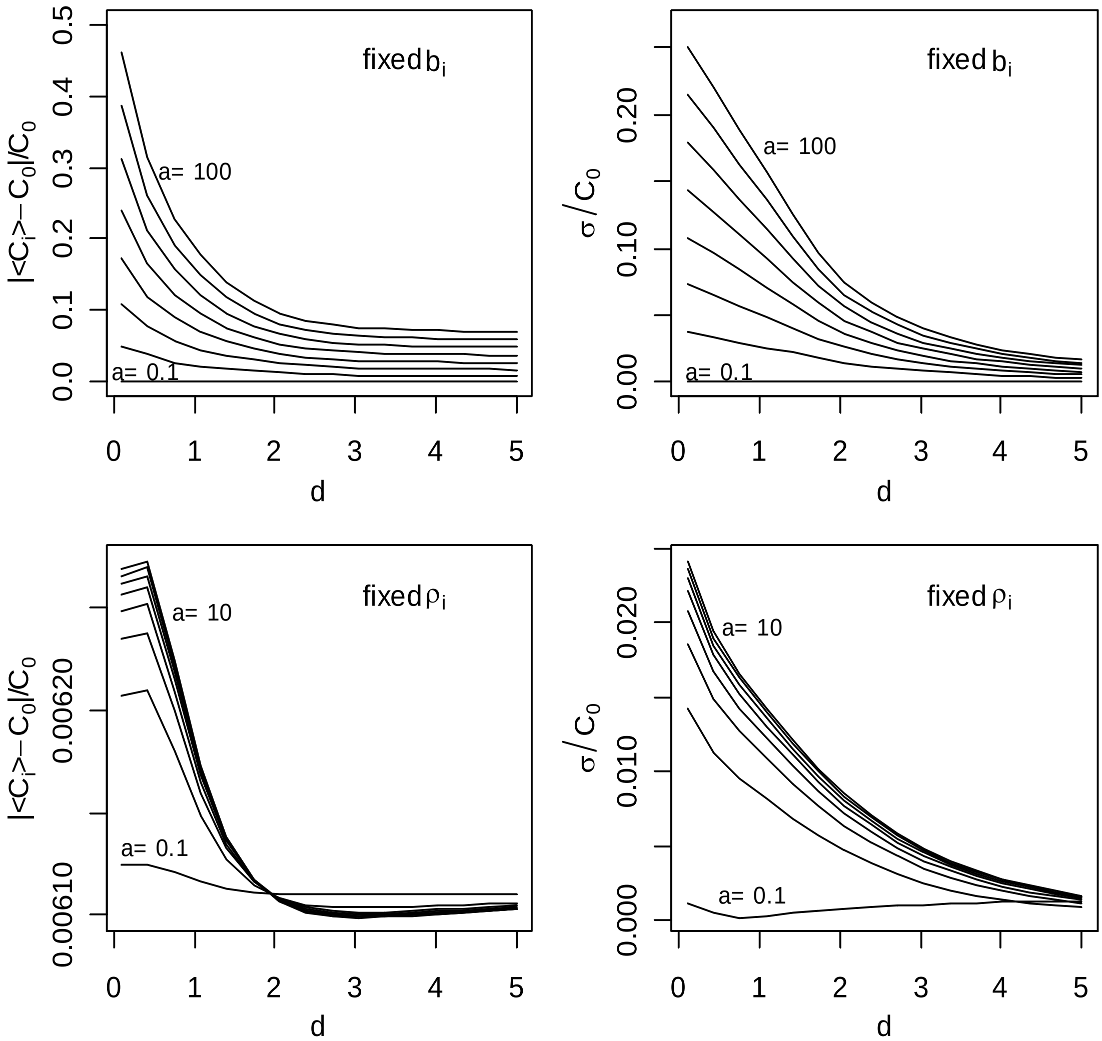

Figure 2 : Left hand diagrams give the precision and right hand diagrams bias of the C_0 estimator. Top diagrams are obtained keeping the sequence of monthly inputs constant and varying dispersion and amplitude of the ρ_i sequence. Bottom diagrams are obtained keeping the ρ_i sequence constant and varying dispersion and amplitude of inputs over the period. *a* and *d* are the parameters used in Eq.(11).

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